Prediktivno upravljanje primjenom matematičkog programiranja

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Outline

- Application Examples
- Predictive Control of Linear Systems
  - Receding Horizon Control
  - Multi-parametric Programming
- Optimal Control of Piecewise Affine Systems
  - Multi-parametric Integer Programming
  - Dynamic Programming based algorithm
- Application Examples revisited
- Is that all?

Application Example 1: Adaptive Cruise Control

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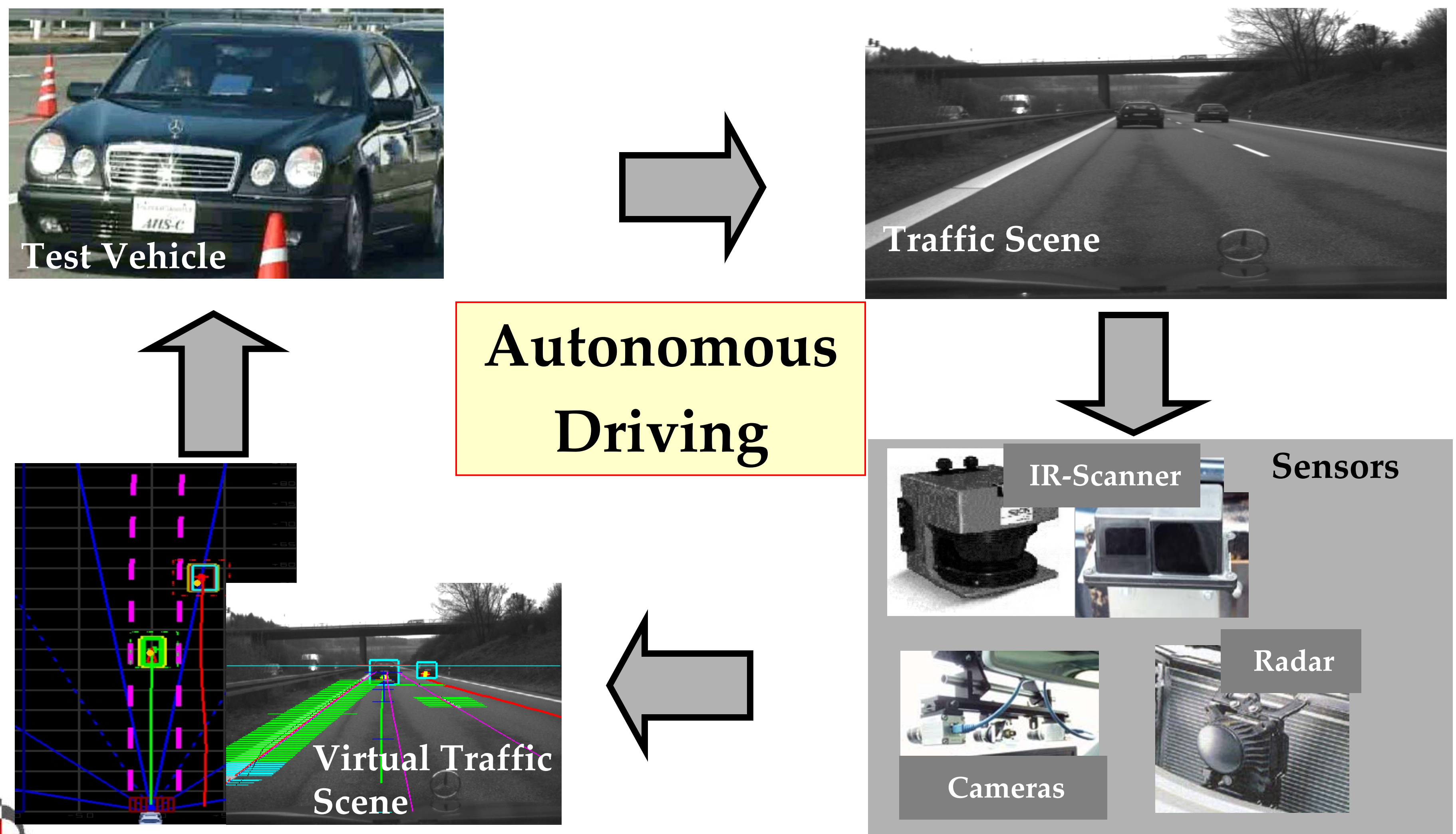


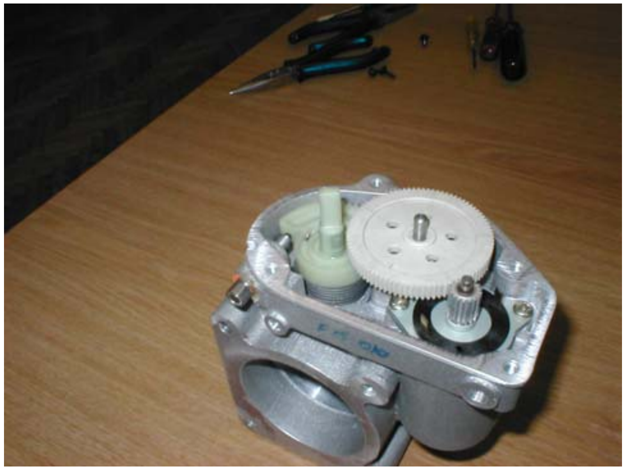
Sensor Fusion

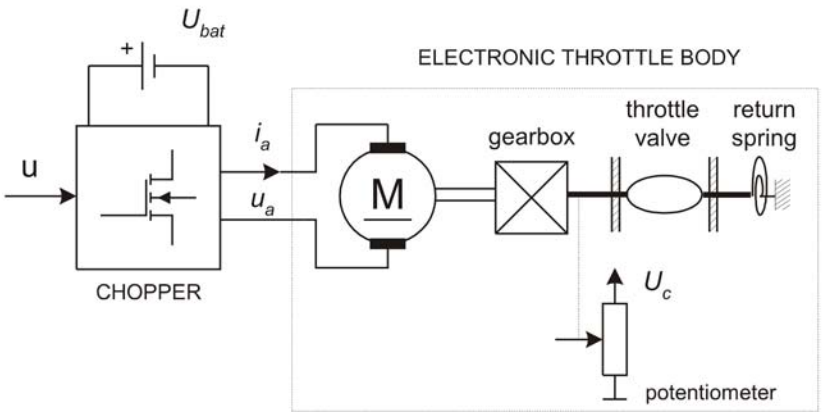
RIC/AA, RIC/AP

BRAKE DISABLED

T [s]	:	0.60
dT [ms]	:	52
FrameNr	:	10990
IB	:	0
A [m/s2]	:	0.429
M [Nm]	:	0.0
V [km/h]	:	103.00
LW [G]	:	69
Y [G/s]	:	-0.860

Application Example 2 – Electronic Throttle



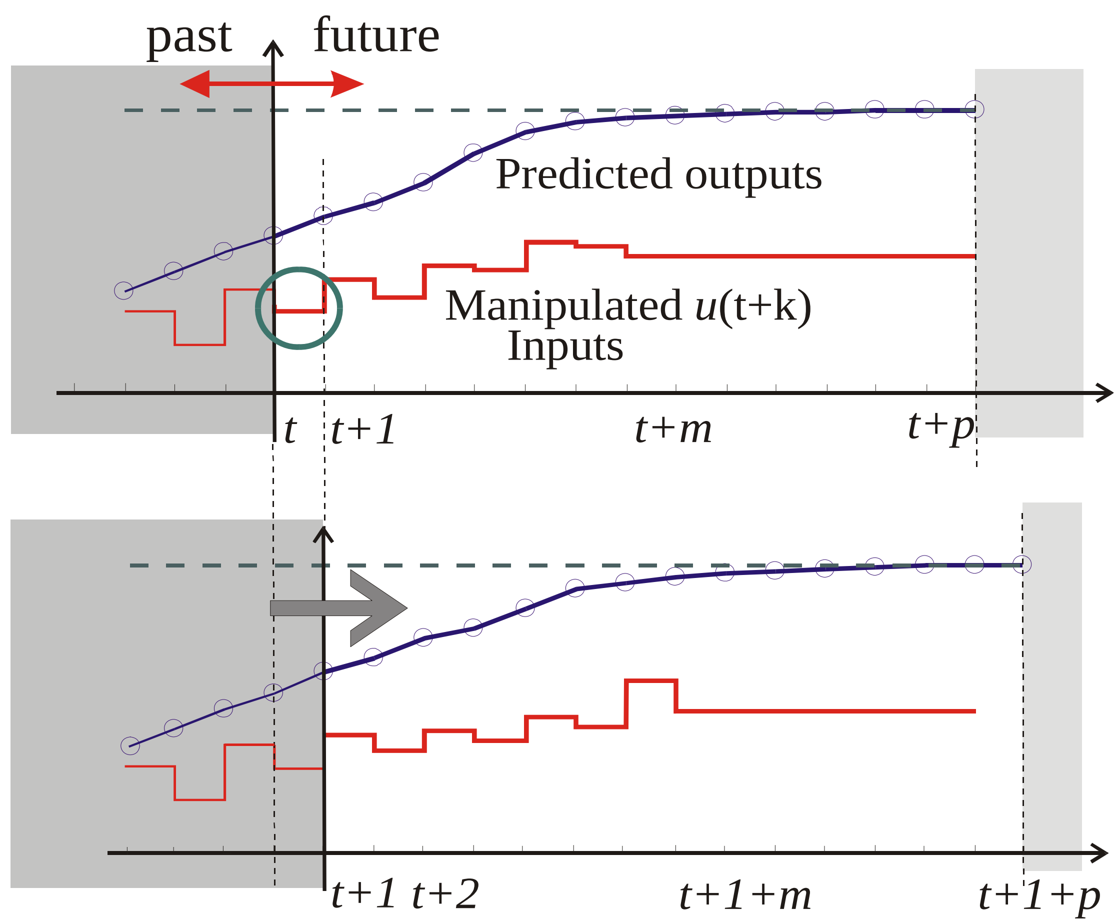


Main challenges

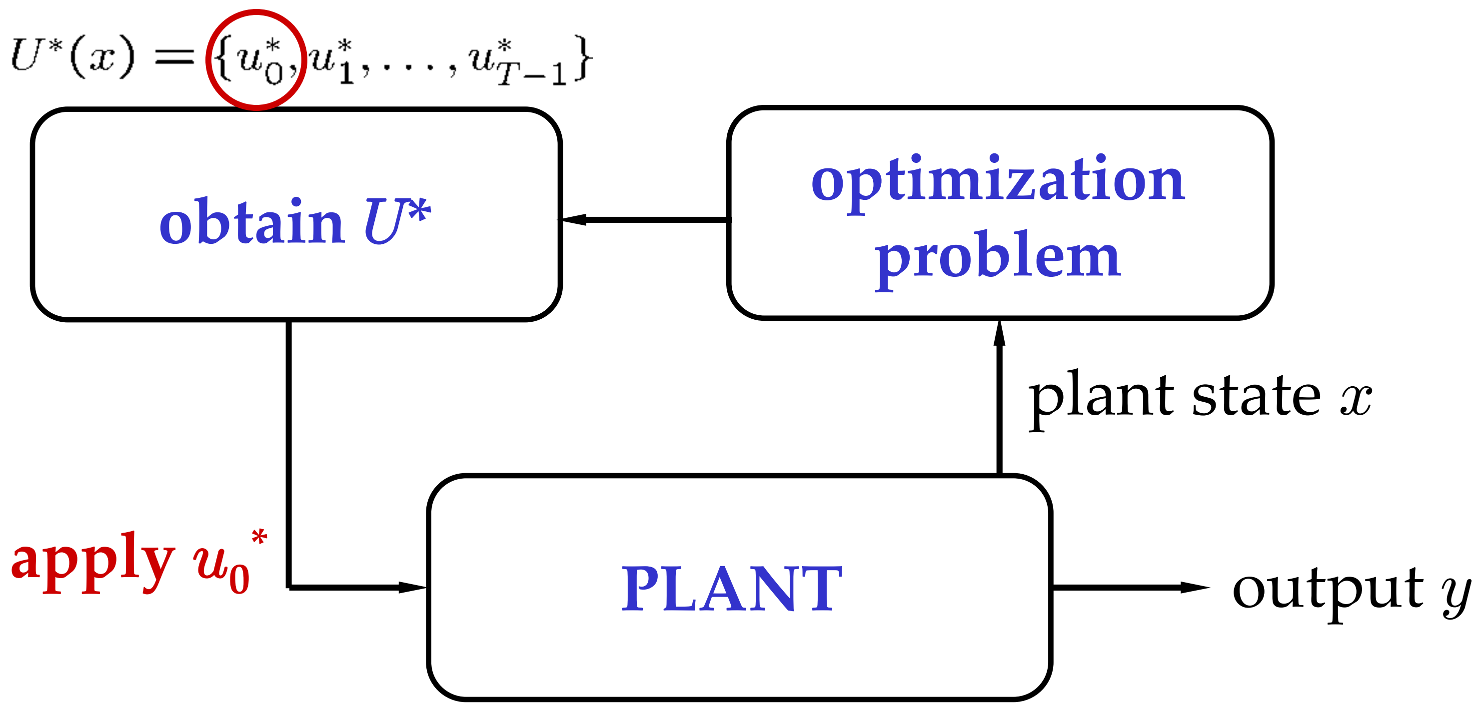
- Friction (gearbox)
- Limp-Home nonlinearity (return spring)
- Constraints
- Quantization (dual potentiometer + A/D)
- Sampling Time : 5 msec

Model Predictive Control (MPC)

- Determine/measure x(t)
- Use **model** to **predict** future behaviour of the system
- Compute optimal sequence of inputs over horizon
- Implement first **control** input $u(t)$
- Wait for next sampling time; $t := t + 1$



Model Predictive Control
– *On-Line* Optimization



“Limitations” of MPC

On-line application is limited

- to "slow" processes
- to "simple" process
- by available computing power

Optimal Control for Constrained Linear Systems

System

- Discrete ***Linear*** Dynamics $x(k+1) = Ax(k) + Bu(k)$
- Constraints on the state $x(k) \in \mathcal{X}$
- Constraints on the input $u(k) \in \mathcal{U}$

$$\left.\begin{array}{l} x(k) \in \mathcal{X} \\ u(k) \in \mathcal{U} \end{array}\right\} C^x x_k + C^u u_k \leq C^0$$

Objectives

- Optimal Performance
- Stability (feedback is stabilizing)
- Feasibility (feedback exists for all time)

Constrained Finite Time Optimal Control (CFTOC) of Linear Systems

Linear Performance Index, $p \in \{1, \infty\}$

$$J^*(x) := \min_U \|Px_T\|_p + \sum_{k=0}^{T-1} \|Qx_k\|_p + \|Ru_k\|_p$$

Constraints

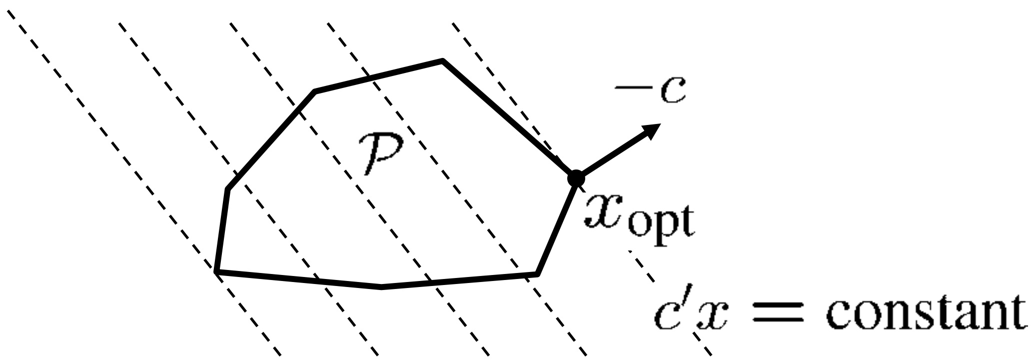
$$\begin{cases} x_0 = x, \\ x_{k+1} = Ax_k + Bu_k, \\ C^x x_k + C^u u_k \leq C^0 \end{cases}$$

Algebraic manipulation ➡ Linear Program (LP)

$$U^*(x) = \{u_0^*, u_1^*, \ldots, u_{T-1}^*\}$$

Linear Program

$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax \leq b \end{array}$$



Standard form:
$$\begin{array}{ll} \min & c'x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

Constrained Finite Time Optimal Control (CFTOC) of Linear Systems

Quadratic Performance Index

$$J^*(x) := \min_U \ x_T' P x_T + \sum_{k=0}^{T-1} x_k' Q x_k + u_k' R u_k$$

Constraints

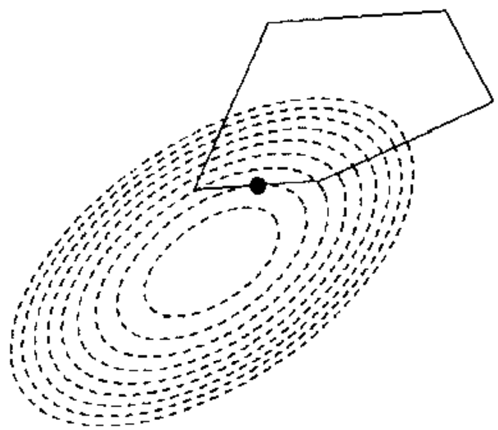
$$\begin{cases} x_0 = x, & P = P' \succeq 0, \\ x_{k+1} = A x_k + B u_k, & Q = Q' \succeq 0, \\ C^x x_k + C^u u_k \leq C^0, & R = R' \succ 0. \end{cases}$$

Algebraic manipulation ➡ Quadratic Program (QP)

$$U^*(x) = \{u_0^*, u_1^*, \ldots, u_{T-1}^*\}$$

Quadratic Program

$$\begin{array}{ll} \min & x'Px + 2c'x + r \\ \text{s.t.} & Ax \le b \end{array}$$



- Convex optimization if $P \succeq 0$
- Very hard problem if $P \not\succeq 0$

Optimization Problem Formulation

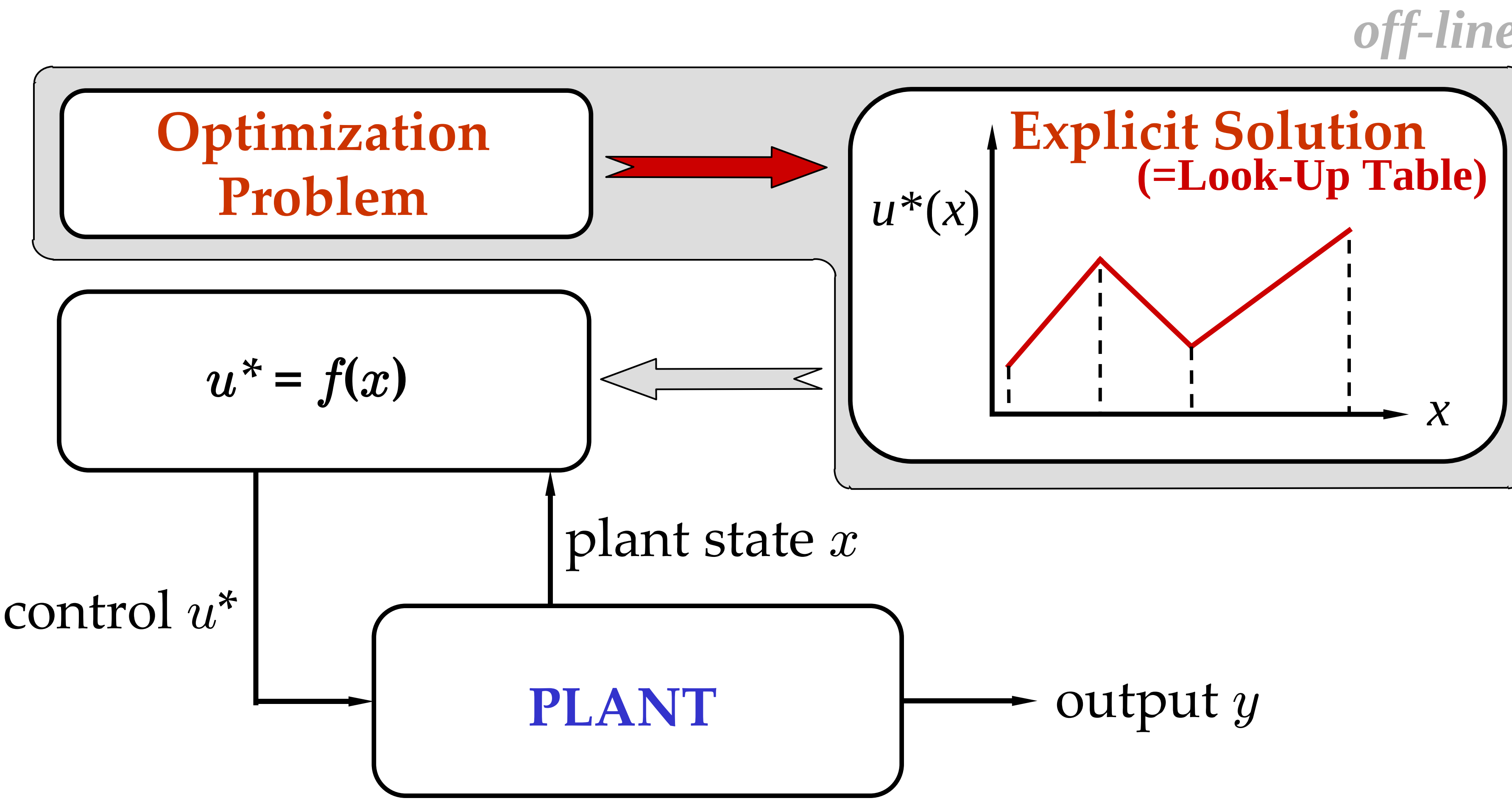
$$\begin{aligned} J^*(x) &= \min_z f(z,x) \\ \text{subj. to} \quad & g(z,x) \leq 0 \end{aligned}$$

z: Decision Variables x: Parameters

Definition:

- Feasible Set $\mathcal{X}_f$ $\mathcal{X}_f = \{ x \in \mathbb{R}^n \mid \exists z \in \mathbb{R}^s, g(z,x) \leq 0 \}$
- Value Function $J^*(x), \quad x \in \mathcal{X}_f$
- Optimizer $z^*(x), \quad x \in \mathcal{X}_f$

Receding Horizon Policy – *Off-Line* Optimization



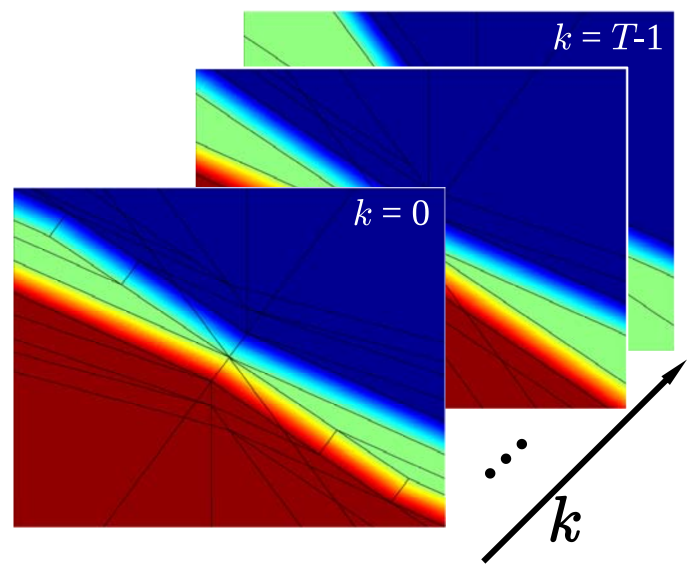
Receding Horizon Policy – *Off-Line* Optimization

- Solve LP/QP ***for all*** x
  via multi-parametric Programming (mp-LP, mp-QP)

Piecewise affine state-feedback law

$$u_k(x) = \begin{cases} F_1^k x + G_1^k & \text{if} \quad H_1^k x \le K_1^k \\ \vdots & \qquad \vdots \\ F_{R_k}^k x + G_{R_k}^k & \text{if} \quad H_{R_k}^k x \le K_{R_k}^k \end{cases}$$

$$\text{for } k = 0, \ldots, T-1$$



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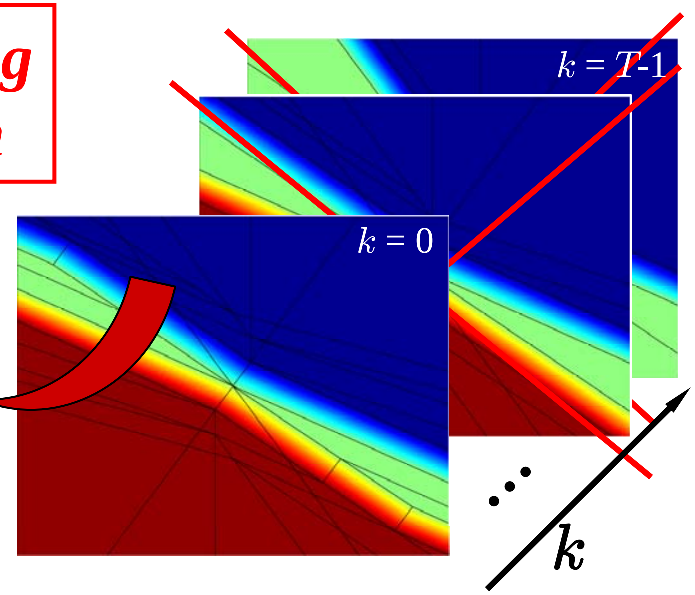
Receding Horizon



$\Rightarrow$ **Look-up table**

Multi-parametric Programming

$$\begin{aligned} J^*(x) &= \min_z f(z,x) \\ \text{subj. to} \quad & g(z,x) \le 0 \end{aligned}$$

z: Decision Variables x: Parameters

- **Conventional math programming:**
  Given x_0, solve optimization problem to obtain $z^*(x_0)$.

- **Multi-parametric Programming:**
  Determine optimizer $z^*(x_0)$ for a range of "parameters" x_0
  $\Rightarrow$ obtain explicit expression for $z^*(x)$.

Example: Solution of mp-QP

$$\begin{array}{ll} \min_z & \frac{1}{2} z' H z \\ \text{subj. to} & Gz \le W + Sx \end{array}$$

$$z \in \mathbb{R}^s, \quad x \in \mathbb{R}^n, \quad W \in \mathbb{R}^c$$

Karush-Kuhn-Tucker (KKT) conditions:

$$\begin{aligned} Hz^* + G'\lambda^* &= 0, \quad \lambda^* \in \mathbb{R}^c \\ \lambda_i^*(G_i z^* - W_i - S_i x) &= 0 \\ \lambda_i^* &\ge 0, \quad i = 1, \ldots, c \end{aligned}$$

Constraint i active $\quad G_i z^* - W_i - S_i x = 0, \ \lambda_i^* \ge 0$

Constraint j inactive $\quad G_j z^* - W_j - S_j x < 0, \ \lambda_j^* = 0$

Example: Solution of mp-QP

$$\begin{array}{ll} \min_z & \frac{1}{2} z' H z \\ \text{subj. to} & Gz \leq W + Sx \end{array}$$

1. Find local linear $z^*(x_0)$

2. Find set where linear $z^*(x_0)$ is valid
   $\rightarrow$ Critical Region

3. Proceed iteratively to find $z^*(x)$, $\forall\, x \in \mathcal{X}_f$

Example: Solution of mp-QP

1. Find local linear $z^*(x)$

- ⇨ solve QP for x_0 to find (z^*,λ^*)
- ⇨ identify active constraints $G_i z^* - W_i - S_i x_0 = 0 \quad (\lambda_i^* \geq 0)$
- ⇨ form matrices $\hat{G}, \hat{W}, \hat{S}$ by collecting active constraints

KKT:

$$\begin{aligned} Hz^* + \hat{G}'\hat{\lambda}^* &= 0 \qquad (1) \\ \hat{G}z^* - \hat{W} - \hat{S}x &= 0 \qquad (2) \end{aligned}$$

From (1) :

$$z^* = -H^{-1}\hat{G}'\hat{\lambda}^*$$

From (2) :

$$\begin{aligned} \hat{\lambda}^*(x) &= -(\hat{G}H^{-1}\hat{G}')^{-1}(\hat{W} + \hat{S}x) \\ z^*(x) &= H^{-1}\hat{G}'(\hat{G}H^{-1}\hat{G}')^{-1}(\hat{W} + \hat{S}x) \end{aligned}$$

➡ In some neighborhood of x_0, λ and z are explicit linear functions of x

Example: Solution of mp-QP

2. Find set where linear $z^*(x)$ is valid – Critical Region

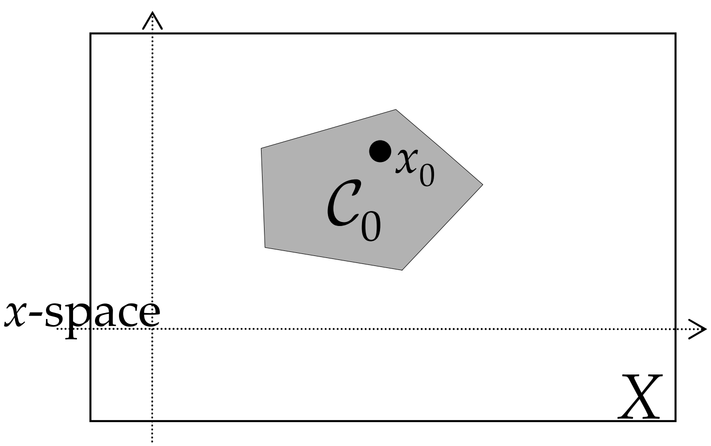
Substitute $\hat{\lambda}^*(x) = -(\hat{G}H^{-1}\hat{G}')^{-1}(\hat{W} + \hat{S}x)$

and

$$z^*(x) = H^{-1}\hat{G}'(\hat{G}H^{-1}\hat{G}')^{-1}(\hat{W} + \hat{S}x)$$

into the constraints:

$$\begin{aligned} \hat{\lambda}^*(x) &\geq 0 \\ Gz^*(x) &\leq W + Sx \end{aligned}$$



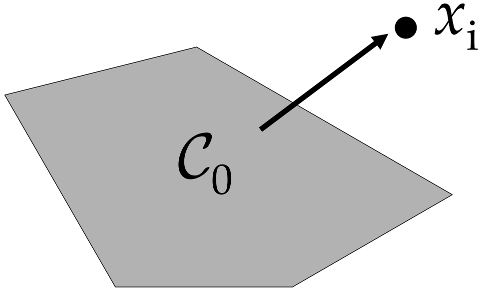
→ Polytopic Critical Region $\mathcal{C}_0$

$$\mathcal{C}_0 = \{x \in \mathbb{R}^n \mid H_0 x \leq K_0\}$$

Example: Solution of mp-QP

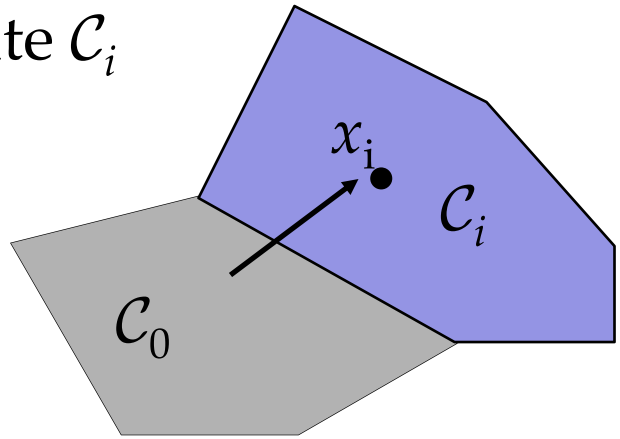
3. Proceed iteratively to find $z^*(x)$, $\forall\, x \in \mathcal{X}_f$

- Pick up new points x_i outside of $\mathcal{C}_0$



- Solve QP for new point x_i
- Extract active constraints and compute $\mathcal{C}_i$

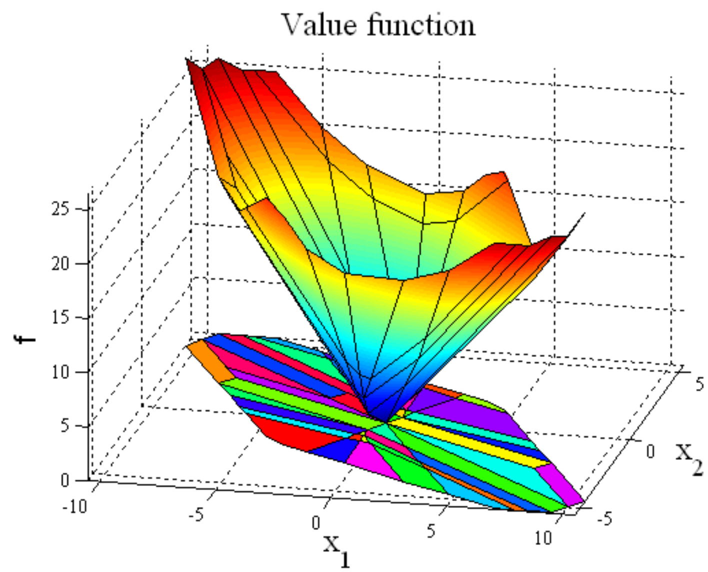
$$\mathcal{C}_i = \{x \in \mathbb{R}^n \mid H_i x \leq K_i\}$$



Characteristics of mp-LP Solution

- The optimizer $z^*(x)$ is continuous and piecewise affine
  (Note: if the optimizer $z^*(x)$ is not unique, there exists one with the above properties)
- The feasible set $\mathcal{X}_f^*$ is convex and partitioned into polyhedral regions
- The value function $J^*(x)$ is convex and piecewise affine

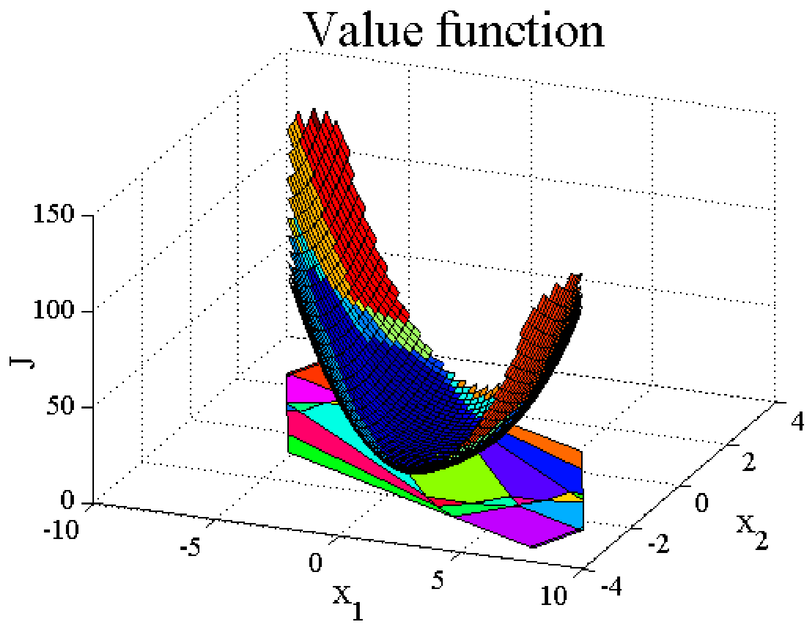
$$z^*(x) = \left\{ \begin{array}{lll} F_1 x + G_1 & \text{if} & H_1 x \leq K_1 \\ \vdots & \vdots & \vdots \\ F_R x + G_R & \text{if} & H_R x \leq K_R \end{array} \right\}$$



Characteristics of mp-QP Solution

- The optimizer $z^*(x)$ is continuous and piecewise affine
- The feasible set $\mathcal{X}_f^*$ is convex and partitioned into polyhedral regions
- The value function $J^*(x)$ is convex, C^1-differentiable and piecewise quadratic

$$z^*(x) = \left\{ \begin{array}{lll} F_1 x + G_1 & \text{if} & H_1 x \le K_1 \\ \vdots & \vdots & \vdots \\ F_R x + G_R & \text{if} & H_R x \le K_R \end{array} \right\}$$



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- Application Examples
- Predictive Control of Linear Systems
  - Receding Horizon Control
  - Multi-parametric Programming
- Optimal Control of Piecewise Affine Systems
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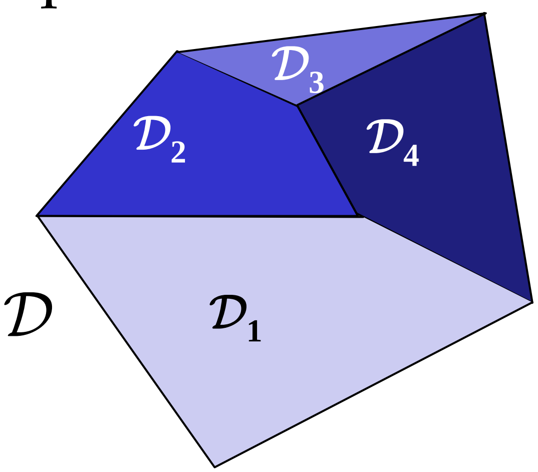
- Application Examples revisited
- Is that all?

Piecewise Affine (PWA) Systems

$$\begin{aligned} x(t+1) &= f_{\text{PWA}}(x(t), u(t)) \\ &= A_i x(t) + B_i u(t) + f_i \quad \text{if} \quad \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}_i \\ & \quad i = 1, \ldots, d \end{aligned}$$

- $\{\mathcal{D}_i\}_{i=1}^{d}$ polyhedral partition of state+input space $\mathcal{D}$
- $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_b}, \;\; n := n_c + n_b$
- $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_b}, \; m := m_c + m_b$
- state and input constraints are included in $\mathcal{D}$



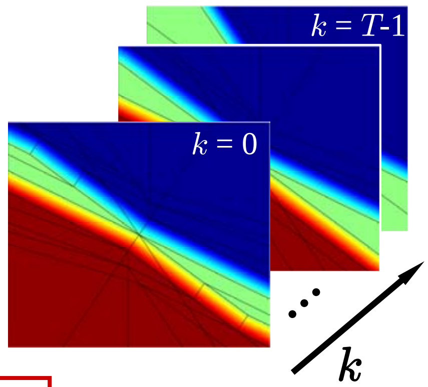
PWA systems are equivalent to other hybrid system classes

CFTOC of PWA Systems – 1/∞-Norm

$$J^*(x(0)) := \min_{U_T} \|Px(T)\|_{\{1,\infty\}} + \sum_{k=0}^{T-1} \|Qx(k)\|_{\{1,\infty\}} + \|Ru(k)\|_{\{1,\infty\}},$$

$$\text{subj. to} \begin{cases} x(t+1) = f_{\text{PWA}}(x(t), u(t)), \\ x(T) \in \mathcal{X}^f. \end{cases}$$

- **Parameters $P, \mathcal{X}^f, T$: influence unknown**
- **Stability: effect of parameters unclear**



Solution to the CFTOC

$$u_k^*(x) = F_i^k x(k) + G_i^k \qquad \text{if} \quad x(k) \in \mathcal{P}_i^k$$

where $\mathcal{P}_i^k$, i=1,…,R_k is a polyhedral partition of the set $\mathcal{X}^k$ of feasible states $x(k)$ at time k with k=0,…,T-1.

CFTOC of PWA Systems – 1/∞-Norm

$$J^*(x(0)) := \min_{U_T} \|Px(T)\|_{\{1,\infty\}} + \sum_{k=0}^{T-1} \|Qx(k)\|_{\{1,\infty\}} + \|Ru(k)\|_{\{1,\infty\}},$$

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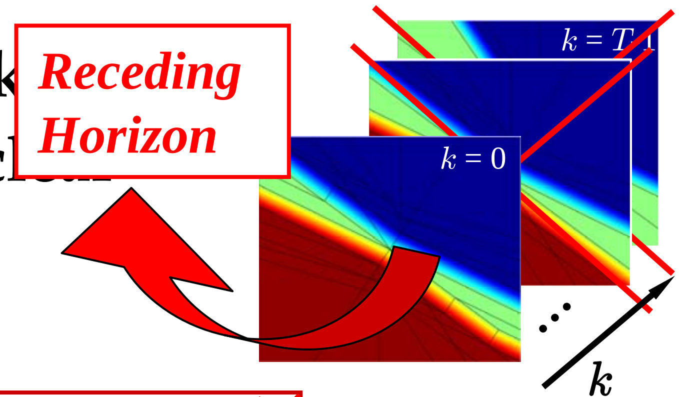
- **Parameters P, $\mathcal{X}^f$, T: influence unk**
- **Stability: effect of parameters uncl**



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CFTOC of PWA Systems – 1/∞-Norm

$$J^*(x(0)) := \min_{U_T} \|Px(T)\|_{\{1,\infty\}} + \sum_{k=0}^{T-1} \|Qx(k)\|_{\{1,\infty\}} + \|Ru(k)\|_{\{1,\infty\}},$$

$$\text{subj. to} \begin{cases} x(t+1) = f_{\text{PWA}}(x(t), u(t)), \\ x(T) \in \mathcal{X}^f. \end{cases}$$

Introduce optimization vector with integer variables

$$\varepsilon = [U_T, \delta, \ldots]^T$$

$$\varepsilon \in \mathbb{R}^{n_c} \times \{0,1\}^{n_b}$$

Multi-**p**arametric **M**ixed-**I**nteger **L**inear **P**rogram

$$\min_{\varepsilon} \; f^T \varepsilon$$
$$\text{subj. to} \; G\varepsilon \le W + Sx$$

(**mp-MILP**)

CFTOC of PWA Systems – Efficient Algorithm

Algorithm based on

- **Dynamic programming** recursion
- Multi-parametric program solver (mp-LP, mp-QP)
- Basic polyhedral manipulation
- Comparison of PWA value functions over polyhedra
- Post processing

Result

- **Exact**
- Globally optimal PWA feedback control law
- Look-up table

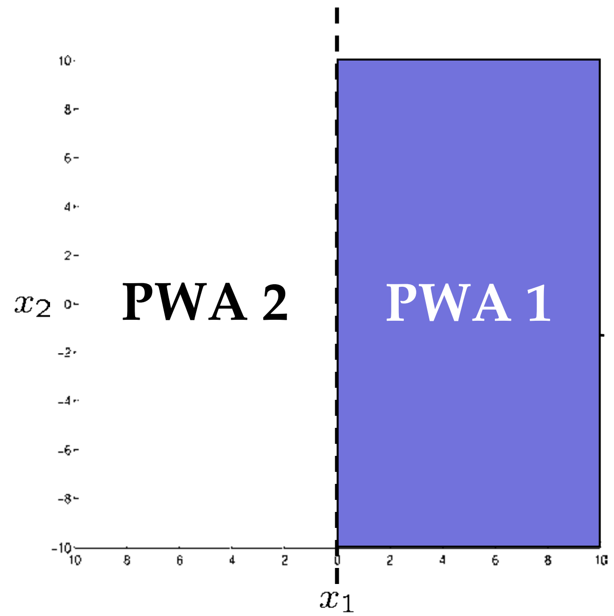
PWA system – Example

$$\min_U \; \|Px_T\|_\infty + \sum_{k=0}^{T-1} \|Qx_k\|_\infty + \|Ru_k\|_\infty$$

$$\begin{cases} x(t+1) & = & 0.8 \begin{bmatrix} \cos\alpha(t) & -\sin\alpha(t) \\ \sin\alpha(t) & \cos\alpha(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ \alpha(t) & = & \begin{cases} \pi/3 & \text{if } [\,1 \;\; 0\,]\,x(t) \geqslant 0 \\ -\pi/3 & \text{if } [\,1 \;\; 0\,]\,x(t) < 0 \end{cases} \\ x(t) & \in & [\,-10, \;\; 10\,] \times [\,-10, \;\; 10\,] \\ u(t) & \in & [\,-1, \;\; 1\,] \end{cases}$$

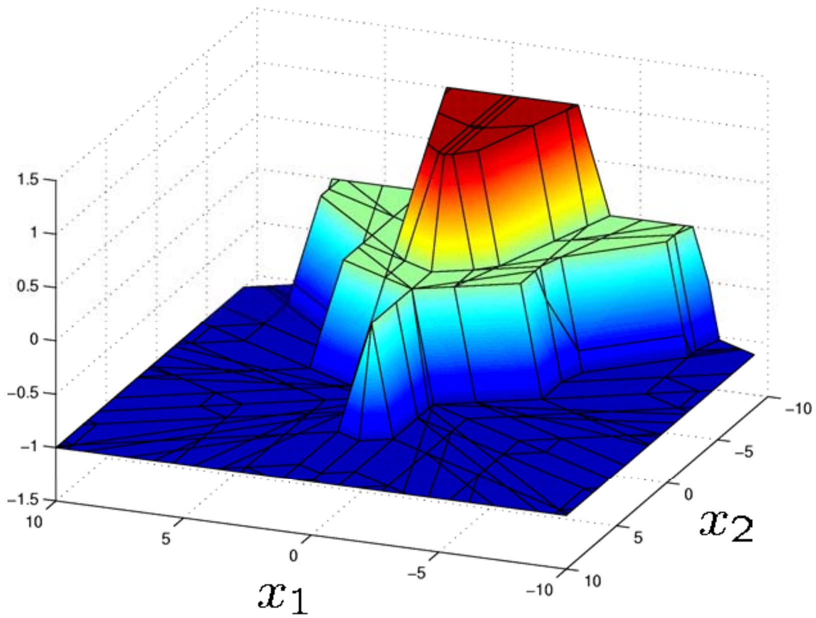
$$P = 0,\; Q = I,\; R = 1$$

$$\mathcal{X}^f = \mathbb{R}^2, \qquad T = \text{free}$$

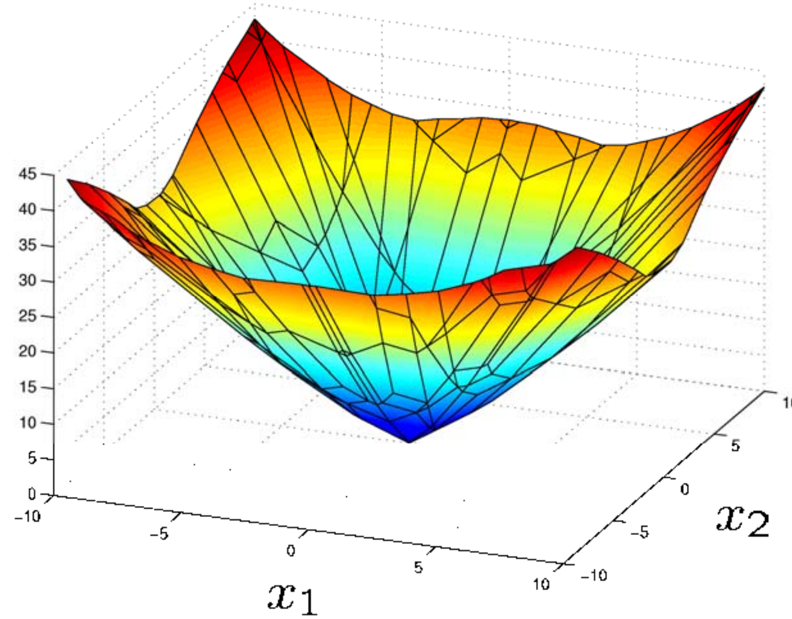


CFTOC Solution
- After 11 (or more) Steps

Optimal Control $u^*(t)$



Value Function J^*

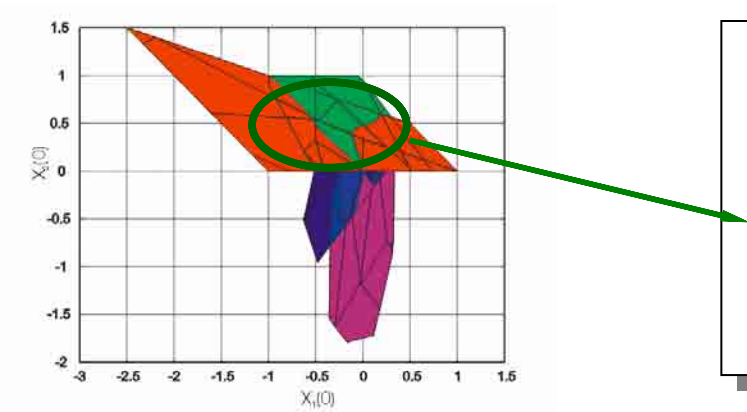


PWA Systems – Linear vs. Quadratic Cost

The solution to the optimal control problem is a ***time varying*** PWA state feedback control law of the form

$$u^*(x) = \begin{cases} F_1 x + G_1 & \text{if} \quad x \in \mathcal{C}_1 \\ \vdots & \vdots \quad \vdots \\ F_R x + G_R & \text{if} \quad x \in \mathcal{C}_R \end{cases}$$

$\{\mathcal{C}_i\}_{i=1}^R$ is a partition of the set of feasible states $x(k)$.



- $\| Qx \|_{\{1,\infty\}}$ $\mathcal{C}_i$ is polyhedral
- $x'Qx$ $\mathcal{C}_i$ is bounded by quadratic functions

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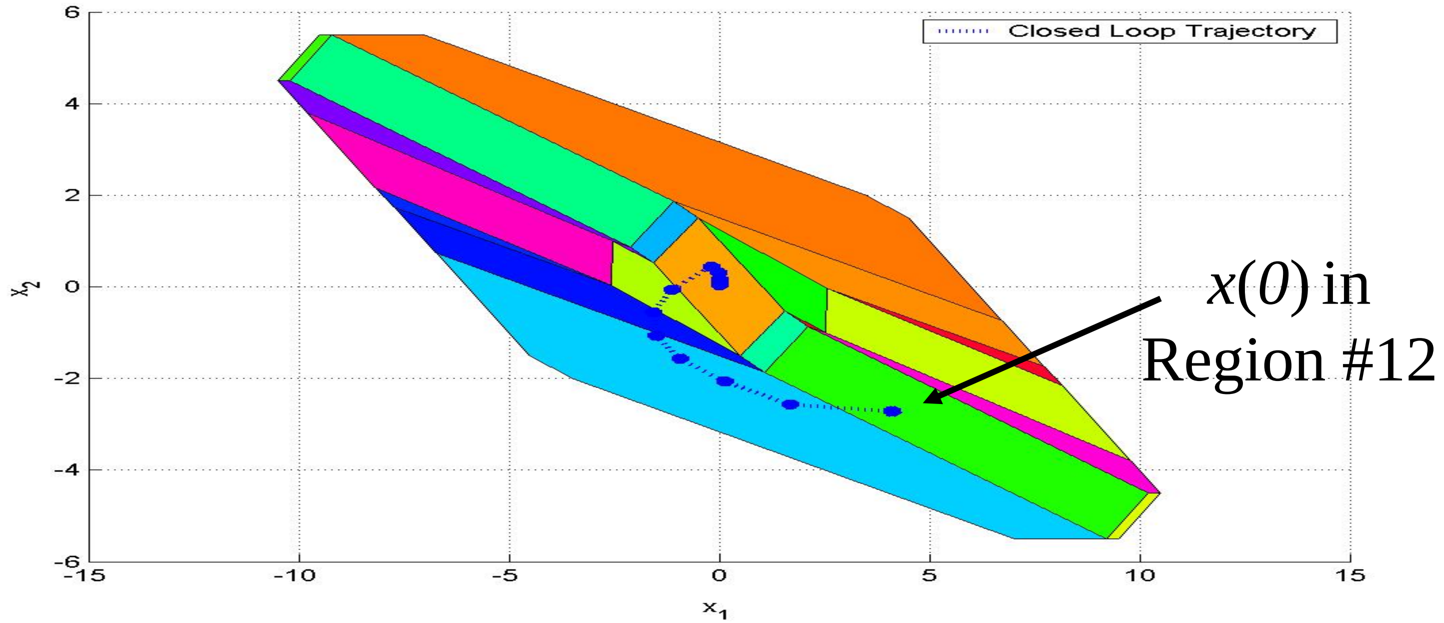
Why Compute an Explicit Solution?

1. Understand the Controller

- Visualization
- Analysis



Why Compute an Explicit Solution?

2. Fast Implementation

$$u(x) = F_r x + G_r, \quad \text{if } H_r x \le K_r$$

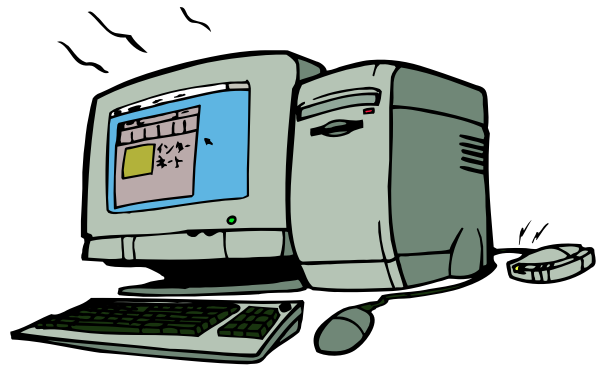
versus

$$\begin{aligned} J_N^*(x(0)) \quad &= \min_{u_0,\dots,u_{N-1}} \left\{ \sum_{k=0}^{N-1} \left(u_k' \mathcal{R} u_k + x_k' \mathcal{Q} x_k \right) + x_N' \mathcal{Q}_f x_N \right\}, \\ \text{subj. to} \quad & x_k \in \mathbb{X}, \qquad k \in [1,\dots,N], \\ & u_k \in \mathbb{U}, \qquad k \in [0,\dots,N-1], \\ & x_{k+1} = A x_k + B u_k, \\ & \mathcal{Q} \succeq 0, \qquad \mathcal{Q}_f \succeq 0, \qquad \mathcal{R} \succ 0. \end{aligned}$$

Why Compute an Explicit Solution?

3. Cheap Implementation

versus



~$10

(Look-Up-Table & µP)

~$10000

(PC & CPLEX)

Application Example 1: Adaptive Cruise Control

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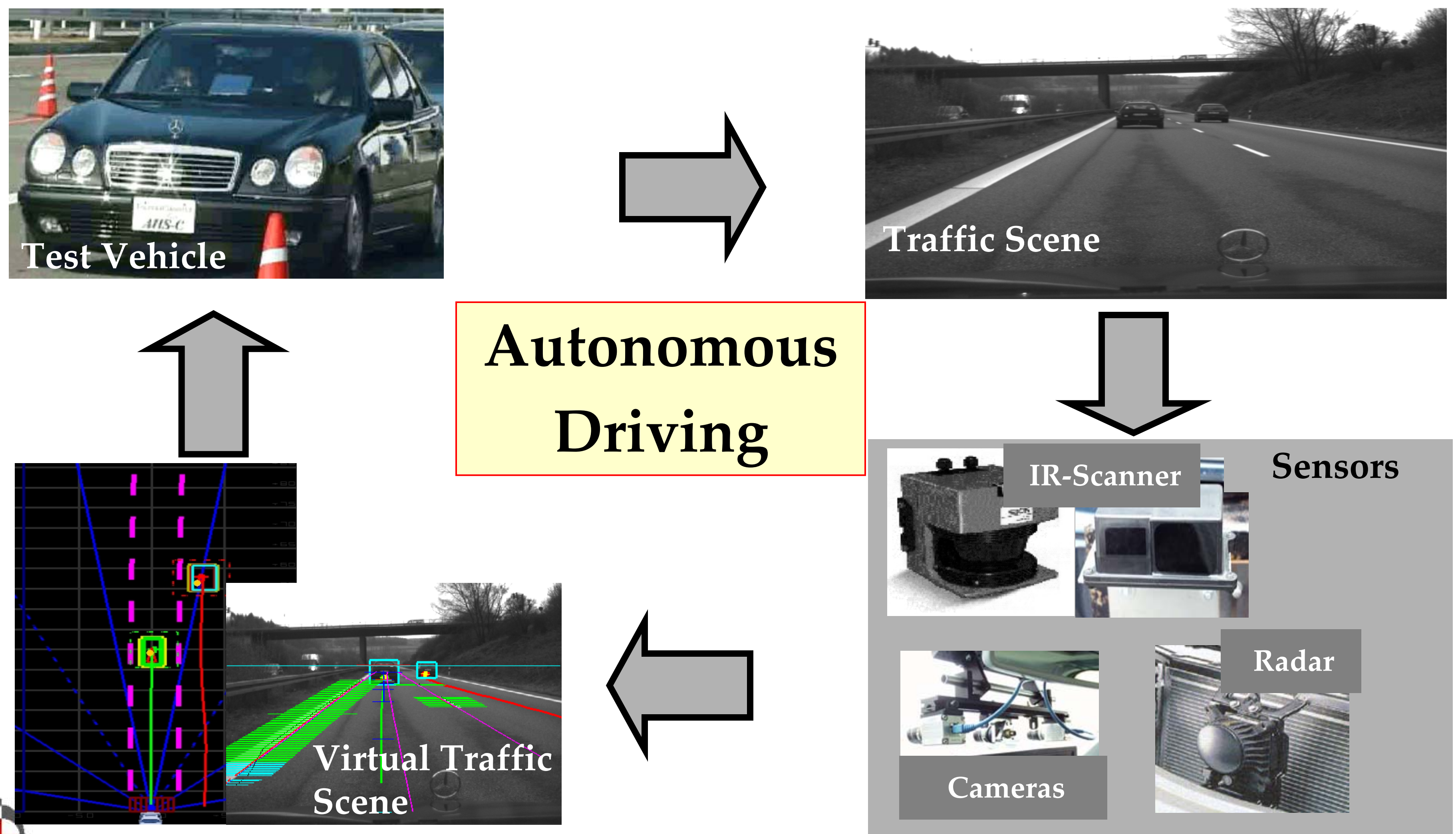

Application Example 1: Adaptive Cruise Control

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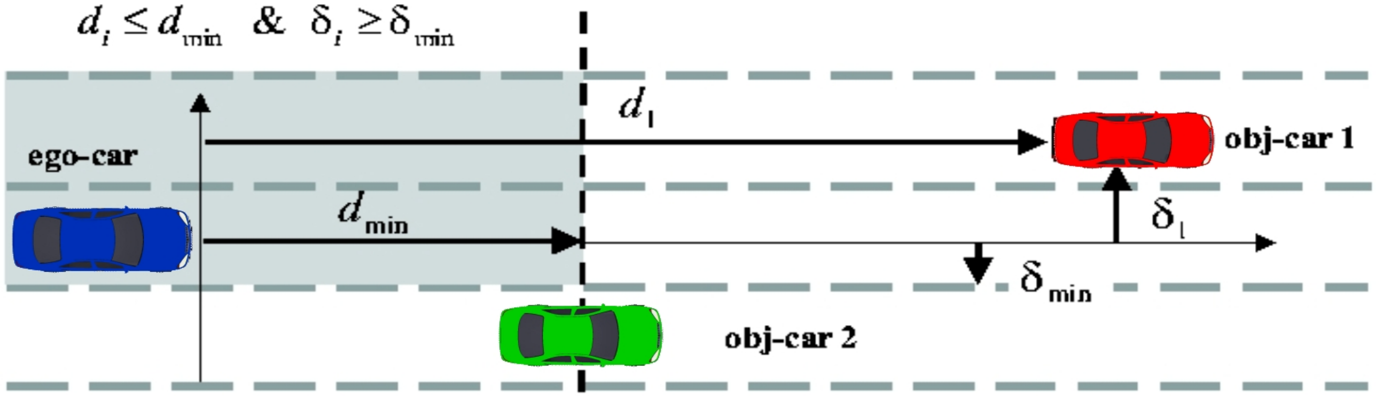
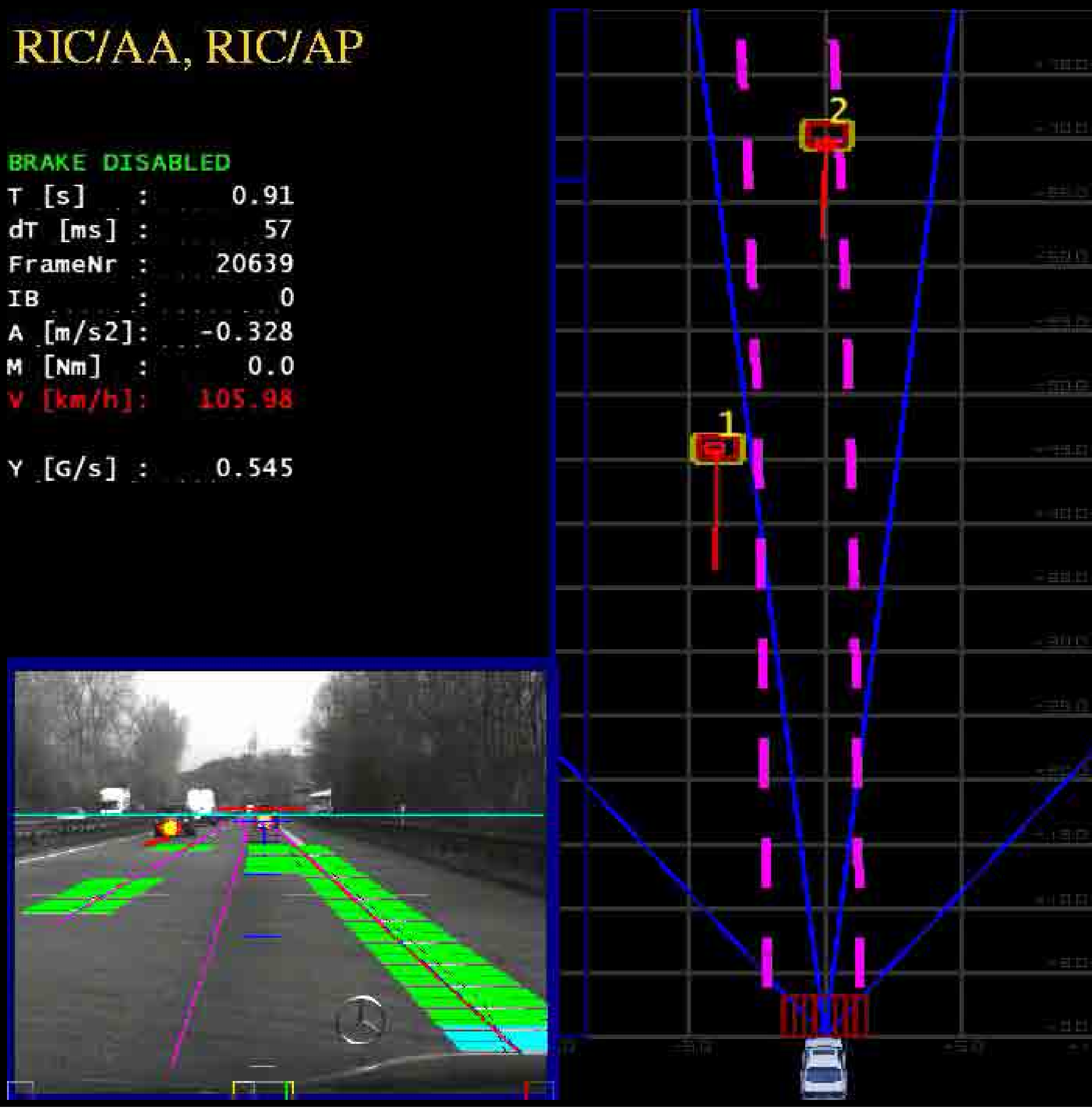


Objectives

- track reference speed
- respect minimum distance
- do not overtake on the right side of a neighboring car
- consider all objects on all lanes
- consider cost over future horizon

Experimental Results

Vref=108km/h



dmin=40m

Application Example 2 – PWA Model of the Throttle

- State vector $x = [\omega_m^*, \theta]'$
- Friction: 5 affine parts
- Limp-Home: 3 affine parts
- Zero Order Hold

15 (discrete time) PWA dynamics

$$\begin{aligned} x(t+1) &= f_{\text{PWA}}(x(t), u(t)) \\ &= A_i x(t) + B_i u(t) + f_i \quad \text{if} \quad H_i x(t) + L_i u(t) \le K_i \\ & \quad i = 1, \dots, 15 \end{aligned}$$

Application Example 2 – Throttle Results

Cost

$$\min_{U} \; \|P(y(T) - r(T))\|_2^2 + \sum_{k=0}^{T-1} \|Q(y(k) - r(k))\|_2^2 + \|R \cdot \Delta u(k)\|_2^2$$

System

$$T_s = 5ms,\ T = 5,\ y(k) = \theta(k),\ \Delta u(k) = u(k) - u(k-1)$$

extended state vector $\bar{x}(k) = [\omega_m(k),\ \theta(k),\ u(k-1),\ r(k)]'$

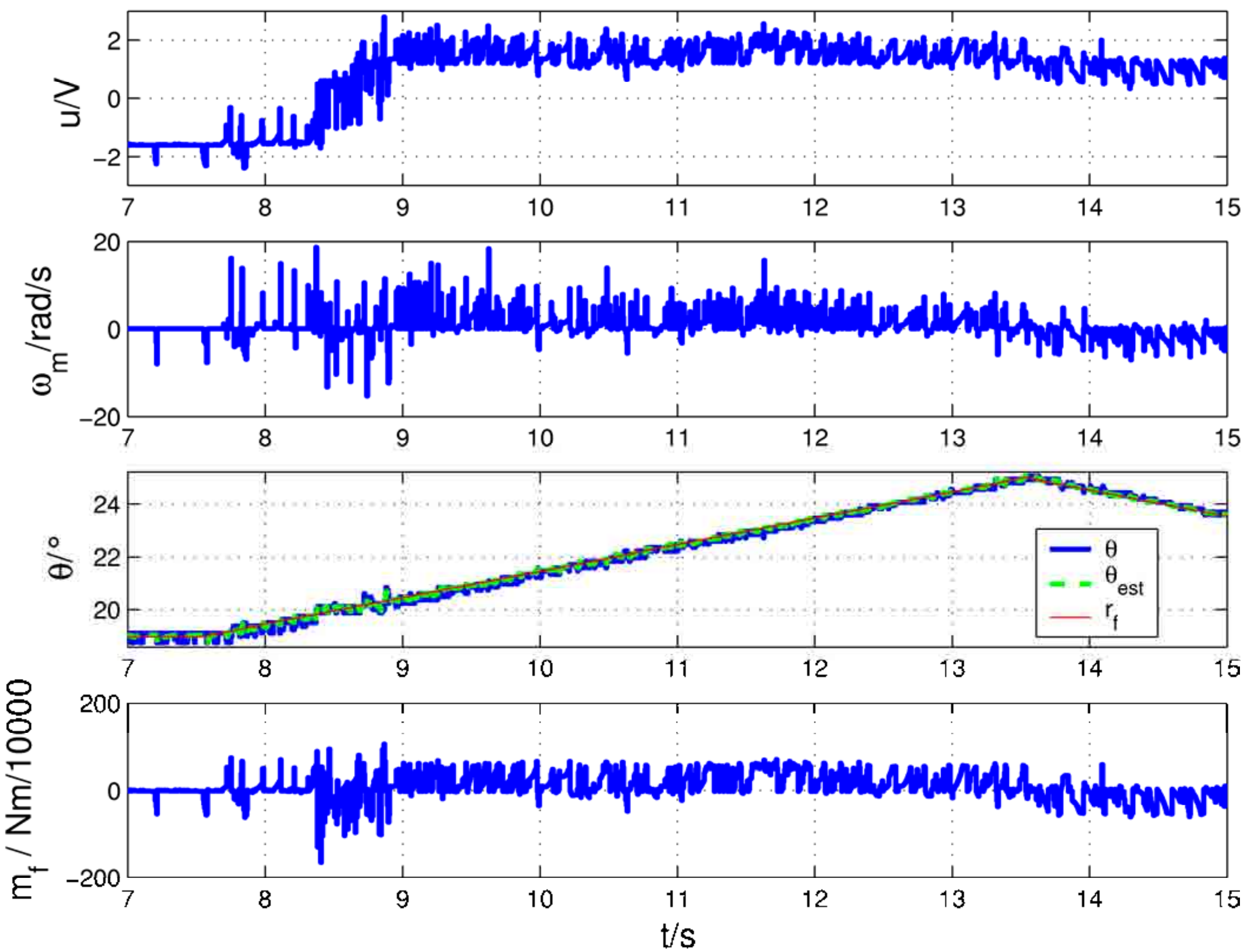
Constraints

$$i_a(k) \in [-3,\ 3],\ \omega_m(k) \in [-300,\ 300]$$
$$\theta(k) \in [11,\ 90],\ u(k) \in [-5,\ 5]$$
$$\Delta u(k) \in [-5,\ 5],\ r(k) \in [11,\ 90]$$

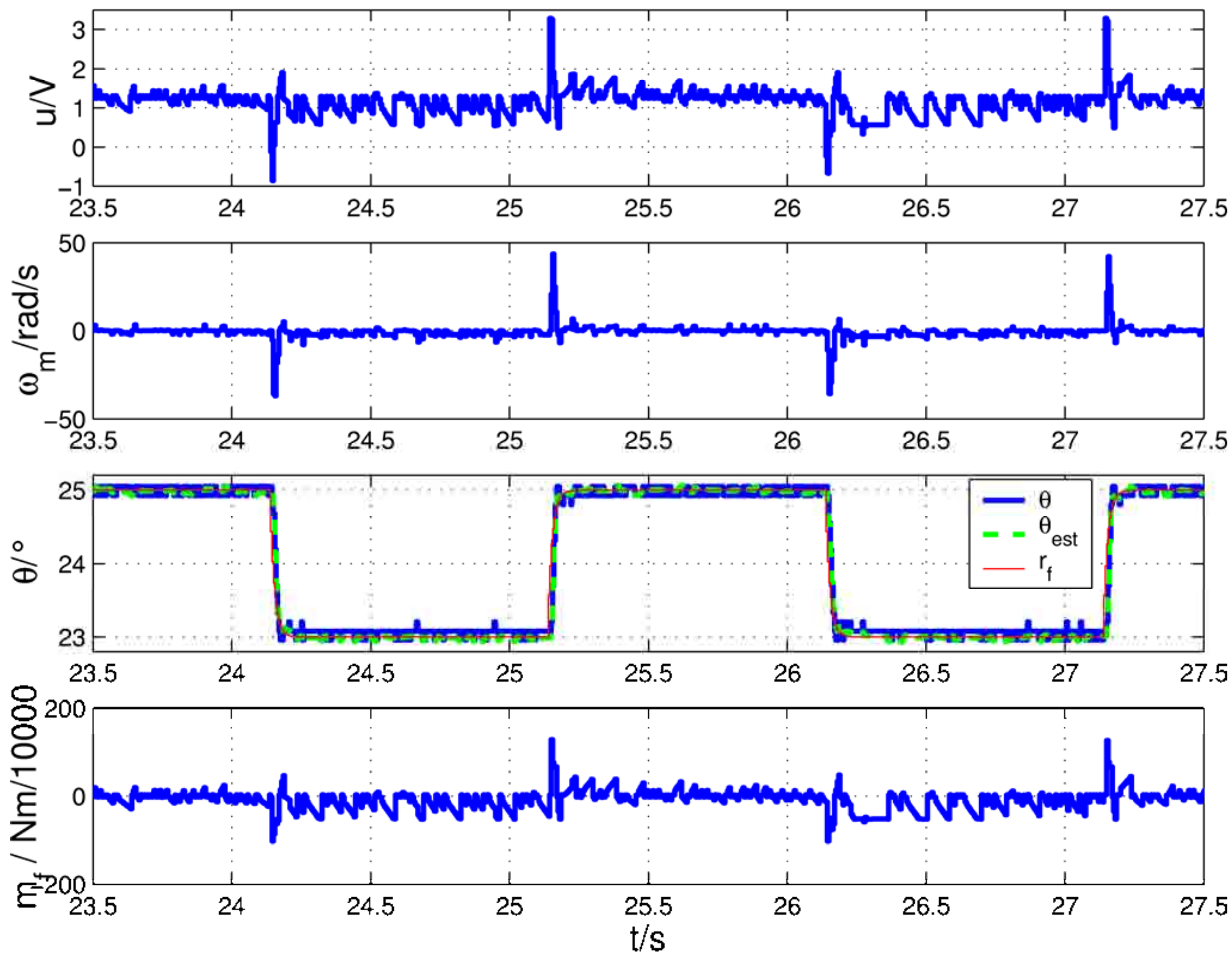
Weights

$$P = 2,\ Q = 1.5,\ R = 1.5$$

Application Example 2 – Throttle Results



Application Example 2 – Throttle Results



Outline

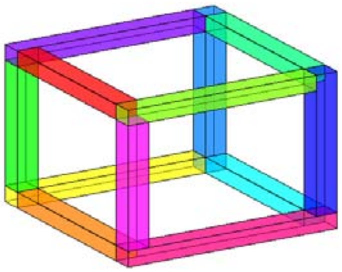
- Application Examples
- Predictive Control of Linear Systems
  - Receding Horizon Control
  - Multi-parametric Programming
- Optimal Control of Piecewise Affine Systems
  - Multi-parametric Integer Programming
  - Dynamic Programming based algorithm
- Application Example revisited
- Is that all?

There is much more

- Exploiting the solution structure
- Exploiting the problem structure
- Infinite Horizon ($T \to \infty$)
- Low complexity control
- Stability & feasibility analysis
- ...

Multi Parametric Toolbox

- All results and plots were obtained with the MPT toolbox



http://control.ethz.ch/~mpt

- MPT is a MATLAB toolbox that provides efficient code for
  - (Non)-Convex Polytope Manipulation
  - Multi-Parametric Programming
  - Control of PWA and LTI systems